CS103 Handout 06
Fall 2013 September 30, 2013

Discussion Problems 1

Some of these questions from Maurizio Caligaris

Every week, we will provide a handout of discussion problems that we hope will help you get a better understanding of the concepts we're currently covering. We'll distribute solutions to these problems in the middle of the week and the TAs will work through them in the weekly recitation sections. Although we won't collect these for a grade, you are **strongly encouraged** to work on them, since they'll help prepare you for the problem sets and will (hopefully!) elucidate some of the more nuanced points we've covered.

Problem One: Identity Elements

Recall from lecture that a binary operator ★ has identity element z iff for any a:

$$a \bigstar z = z \bigstar a = a$$

Not all binary operators have identity elements, though many do. However, it's impossible for a binary operator to have several identity elements. Prove that if ★ is a binary operation with identity elements z_1 and z_2, then $z_1 = z_2$. This is an example of a *uniqueness proof*, in which you show that at most one object with a certain property can exist by proving that if there are two objects with a certain description, they must actually be the same object. All we did was give that object two names.

Problem Two: Balls in Bins

Suppose that you have twenty-five balls to place into five different bins. Eleven of the balls are red, while the other fourteen are blue. Prove that no matter how the balls are placed into the bins, there must be at least one bin containing at least three red balls.

Problem Three: Quadratic Equations

A *quadratic equation* is an equation of the form $ax^2 + bx + c = 0$. A *root* of the equation is a real number x satisfying the equation.

Recall from lecture that a rational number is one that can be written as p/q for integers p and q where $q \neq 0$ and p and q have no common divisor other than ± 1.

i. Prove that mn is odd iff m is odd and n is odd.
ii. Prove, by contradiction, that if a, b, and c are odd numbers, then there are no rational numbers x for which $ax^2 + bx + c = 0$. Be sure to explicitly state what assumption you are attempting to contradict. As a hint, if the rational solution is p / q, consider what happens if both p and q are odd and what happens if exactly one of p and q is odd. (Why can't both p and q be even?)

Problem Four: Finding Flaws in Proofs

The following proofs all contain errors that allow them to prove results that are incorrect. For each proof, identify at least one flaw in the proof and explain what the problem is, then give a counterexample that demonstrates why the error occurs. In each case, **make sure you understand what logical error is being made**. The mistakes made here are extremely common.

Theorem: If n is even, then n^2 is odd.
Proof: By contradiction; assume that n is odd but that n^2 is even. Since n is odd, $n = 2k + 1$ for some integer k. Thus $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which is odd. This contradicts our earlier claim that n^2 is even. We have reached a contradiction, so our initial assumption was wrong. Thus if n is even, n^2 is odd. ■

Theorem: For all sets A and B, $A \cup B = A$.
Proof: By contradiction; assume that for all sets A and B, $A \cup B \neq A$. So consider $A = \emptyset$ and $B = \emptyset$; then $A \cup B = A$. This contradicts our earlier claim that $A \cup B \neq A$ for all A and B. We have reached a contradiction, so our initial assumption was wrong. Thus for any sets A and B, $A \cup B = A$. ■

Theorem: If $C \subseteq A \cup B$, then $C \subseteq A$.
Proof: By contrapositive. We prove that if C is not a subset of $A \cup B$, then it is not a subset of A. Since C is not a subset of $A \cup B$, there is some $x \in C$ such that $x \notin A \cup B$. Since $x \notin A \cup B$, $x \notin A$ and $x \notin B$. Thus $x \in C$ but $x \notin A$, and so C is not a subset of A. ■